The 2D incremental ray-propagator

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Abstract

We propose an incremental type of ray-propagator which can be used to identify the typical information that respectively the point-source and telescopic-source solutions carry with them regarding the 2D medium parameters (assuming a smooth earth model). Both forward time propagation and reverse-time propagation are considered. Moreover, a source-perturbed version of the incremental system is used to analyze the velocity stretching problem, i.e. how to estimate more proper interval velocities from either time-migration or stacking velocities.

Introduction

Paraxial ray-tracing in a smooth model is a well-established technique (Červený, 2001). This paper makes an attempt to, in a rather naïve way, to unravel the information content carried by the fundamental point-source and telescopic-source solutions by considering a so-called incremental ray-propagator system. The results can also be used to give further insight into the ray quantities associated with the surface-to-surface ray propagator formulation of Bortfeld (1989) and generalized stacking formulations like the Common Reflection Surface (CRS) technique (Jäger et al., 2001).

By considering a special version of the incremental propagator system, the problem of velocity stretching can also be easily investigated. The velocity stretching problem means how to map stacking velocities or time-migration velocities back to local interval velocities. This procedure was first discussed by Cameron et al. (2007) within the context of prestack time migration. In this paper we also demonstrate that an analogous analysis can be carried out with respect to a generalized stacking procedure such as CRS.

Forward incremental ray-propagator

The starting point is the paraxial dynamic ray-tracing system in ray-centered coordinates, which in a 2D medium reads (Červený, 2001)

$$\frac{dQ}{dT} = v^2 P \quad , \quad \frac{dP}{dT} = -v^{-1} v_{qq} Q \quad , \tag{1}$$

where T is the traveltime along the central ray, v is the medium velocity (measured along the central or reference ray) and v_{qq} is the second derivative of the velocity with respect to the ray centered coordinate q (i.e. along a direction orthogonal to the central ray direction). Taking the time derivative of the above two equations gives

$$\begin{aligned} \frac{d^2Q}{dT} &= 2v\frac{dv}{dT}P + v^2\frac{dP}{dT} = 2v\frac{dv}{dT}P - v v_{qq} Q \quad , \\ \frac{d^2P}{dT} &= -\frac{d}{dT}(v^{-1}v_{qq})Q - v^{-1}v_{qq}\frac{dQ}{dT} \\ &= -v^{-1}v_{qq}\frac{d\ln(v^{-1}v_{qq})}{dT}Q - v v_{qq} P \end{aligned} \tag{2}$$

Consider now two nearby points $\vec{x}_i$ and $\vec{x}_{i+1}$ (not end points) along the central ray representing an incremental traveltime difference δT and introduce the following Taylor expansions to second order in traveltime (employing the notation $Q_j \equiv Q(\vec{x}_j)$ and similarly for P, v and v_{qq})

$$\begin{aligned} Q_{i+1} &\cong Q_i + \frac{dQ_i}{dT}\delta T + \frac{1}{2}\frac{d^2Q_i}{dT^2}\delta T^2 \\ &\cong \left(1 - \frac{1}{2}v_i v_{i,qq}\delta T^2\right)Q_i + \left(v_i^2 \delta T\right)P_i , \\ P_{i+1} &\cong P_i + \frac{dP_i}{dT}\delta T + \frac{1}{2}\frac{d^2P_i}{dT^2}\delta T^2 \\ &\cong \left[-v_i^{-1}v_{i,qq}\delta T\right]Q_i + \left(1 - \frac{1}{2}v_i v_{i,qq}\delta T^2\right)P_i . \end{aligned} \tag{3}$$

In Eq.(3) we have neglected the time derivatives in Eq.(2) and assumed constant values within each time step. As usual practice, the system in Eq.(3) can be conveniently recast in matrix form

$$\begin{bmatrix} Q_{i+1} \\ P_{i+1} \end{bmatrix} = {}_{\uparrow}\Delta\pi_i \begin{bmatrix} Q_i \\ P_i \end{bmatrix} , \tag{4}$$

where ${}_{\uparrow}\Delta\pi_i$ is the incremental ray-propagator matrix given by

$$
{}_{\uparrow}\Delta\pi_i = \begin{bmatrix} {}_{\uparrow}\Delta Q_1 & {}_{\uparrow}\Delta Q_2 \\ {}_{\uparrow}\Delta P_1 & {}_{\uparrow}\Delta P_2 \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{2}v_i v_{i,qq}\delta T^2 & v_i^2\delta T \\ -v_i^{-1}v_{i,qq}\delta T & 1-\frac{1}{2}v_i v_{i,qq}\delta T^2 \end{bmatrix}. \quad (5)
$$

In analogy with the usual definition of the propagator matrix (Červený, 2001), the columns $({}_{\uparrow}\Delta Q_1\ {}_{\uparrow}\Delta P_1)^T$ and $({}_{\uparrow}\Delta Q_2\ {}_{\uparrow}\Delta P_2)^T$ of the matrix ${}_{\uparrow}\Delta\pi_i$ represent the incremental plane-wave (telescopic) and point-source solutions of the dynamical ray-tracing system along the ray that connect the nearby points $\vec{x}_i$ and $\vec{x}_{i+1}$. In the same way, ${}_{\uparrow}\Delta\pi_i$ fulfills the symplectic condition (Červený, 2001) (within 2nd order time perturbation)

$$
({}_{\uparrow}\Delta\pi_i)^T J({}_{\uparrow}\Delta\pi_i) = J \quad , \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad (6)
$$

from which $|{}_{\uparrow}\Delta\pi_i| = 1$.

Time-reversed incremental ray-propagator

The time-reversed version of Eq.(4) will formally read

$$
\begin{bmatrix} Q_i \\ P_i \end{bmatrix} = ({}_{\uparrow}\Delta\pi_i)^{-1}\begin{bmatrix} Q_{i+1} \\ P_{i+1} \end{bmatrix} \equiv {}_{\downarrow}\Delta\pi_i)\begin{bmatrix} Q_{i+1} \\ P_{i+1} \end{bmatrix}. \quad (7)
$$

Note that the equality between the inverse and the time-reversed incremental ray propagator holds because *we are not considering the complete ray propagation from source to receiver*. From the symplectic condition, together with the definition of the ray centered coordinates, we have, (Červený, 2001)

$$
{}_{\downarrow}\Delta\pi_i = {}_{\uparrow}\Delta\pi_i^{-1} = J^T({}_{\uparrow}\Delta\pi_i^T)J = \begin{bmatrix} {}_{\downarrow}\Delta Q_1 & {}_{\downarrow}\Delta Q_2 \\ {}_{\downarrow}\Delta P_1 & {}_{\downarrow}\Delta P_2 \end{bmatrix} = \begin{bmatrix} {}_{\uparrow}\Delta P_2 & -{}_{\uparrow}\Delta Q_2 \\ -{}_{\uparrow}\Delta P_1 & {}_{\uparrow}\Delta Q_1 \end{bmatrix}. \quad (8)
$$

Cascaded solutions

Based on Eqs.(4) and (5), corresponding cascaded solutions can be constructed. Assume a total of N time steps (corresponding to a total traveltime $T = N\Delta T$). Then we can write, in case of forward propagation in time (assuming either point or telescopic source initial condition),

$$
\begin{bmatrix} {}_{\uparrow}Q_f \\ {}_{\uparrow}P_f \end{bmatrix} = ({}_{\uparrow}\Delta\pi_N)\cdots({}_{\uparrow}\Delta\pi_2)({}_{\uparrow}\Delta\pi_1)\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix} = {}_{\uparrow}\pi\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix}, \quad (9a)
$$

where ${}_{\uparrow}\pi$ represents the ray-propagator matrix along the full ray

$$
{}_{\uparrow}\pi = ({}_{\uparrow}\Delta\pi_N)\cdots({}_{\uparrow}\Delta\pi_2)({}_{\uparrow}\Delta\pi_1) = \begin{bmatrix} {}_{\uparrow}Q_1 & {}_{\uparrow}Q_2 \\ {}_{\uparrow}P_1 & {}_{\uparrow}P_2 \end{bmatrix}. \quad (9b)
$$

The telescopic and point-source components of the propagator matrix are found to be

$$
{}_{\uparrow}Q_1 = 1-\sum_{i=1}^{N}\left\{\sum_{k=i}^{N}v_k^2\delta T - \frac{3v_i^2\delta T}{2}\right\}v_i^{-1}v_{i,qq}\delta T \cong 1 - T^2v_{rms}^2(0,T)\left\langle v^{-1}v_{qq}\right\rangle + T\left\langle v^{-1}v_{qq}\right\rangle_w , \quad (10)
$$

$$
{}_{\uparrow}P_1 = -\sum_{i=1}^{N}v_i^{-1}v_{i,qq}\delta T \cong -T\left\langle v^{-1}v_{qq}\right\rangle \quad (11)
$$

$$
{}_{\uparrow}Q_2 = \sum_{i=1}^{N}v_i^2\delta T \cong \int_{T'=0}^{T}v^2(T')dT' = Tv_{rms}^2 , \quad (12)
$$

$$
{}_{\uparrow}P_2 = 1-\sum_{i=1}^{N}\left\{\sum_{k=1}^{i}v_k^2\delta T - \frac{3v_i^2\delta T}{2}\right\}v_i^{-1}v_{i,qq}\delta T \cong 1 - T\left\langle v^{-1}v_{qq}\right\rangle_w , \quad (13)
$$

in which we have employed the time averages

$$
\left\langle v^{-1}v_{qq}\right\rangle = \frac{1}{T}\int_{T'=0}^{T}v^{-1}(T')v_{qq}(T')dT' \qquad \left\langle v^{-1}v_{qq}\right\rangle_w = \frac{1}{T}\int_{T'=0}^{T}T'v_{rms}^2(0,T')v^{-1}(T')v_{qq}(T')dT'. \quad (14)
$$

In Eqs.(10)-(13), the notation $v_{rms}(T_i,T_f)$ implies an rms-velocity calculated along the central ray between propagation times T_i and T_f. We also adopted the simplifying notation $v_{rms}(0,T) = v_{rms}$.

The corresponding cascaded system in case of time-reversed propagation reads

$$
\begin{bmatrix} {}_{\downarrow}Q_f \\ {}_{\downarrow}P_f \end{bmatrix} = ({}_{\downarrow}\Delta\pi_1)\cdots({}_{\downarrow}\Delta\pi_{N-1})({}_{\downarrow}\Delta\pi_N)\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix} = J^T({}_{\uparrow}\Delta\pi_1)^T\cdots({}_{\uparrow}\Delta\pi_{N-1})^T({}_{\uparrow}\Delta\pi_N)^T J\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix} = \left(J^T{}_{\uparrow}\pi J\right)^T\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix} = {}_{\downarrow}\pi\begin{bmatrix} Q_{ini} \\ P_{ini} \end{bmatrix}, \quad (15)
$$

which, in combination with Eq.(9a), gives the well-known relationships between forward and reverse time-propagated ray matrices (by analogy with Eq.(8)) (Červený, 2001)

$$
{}_{\downarrow}\pi = \begin{bmatrix} {}_{\downarrow}Q_1 & {}_{\downarrow}Q_2 \\ {}_{\downarrow}P_1 & {}_{\downarrow}P_2 \end{bmatrix} = J^T{}_{\uparrow}\pi^T J = J^T\begin{bmatrix} {}_{\uparrow}Q_1 & {}_{\uparrow}Q_2 \\ {}_{\uparrow}P_1 & {}_{\uparrow}P_2 \end{bmatrix}^T J \equiv \begin{bmatrix} {}_{\uparrow}P_2 & -{}_{\uparrow}Q_2 \\ -{}_{\uparrow}P_1 & {}_{\uparrow}Q_1 \end{bmatrix}. \quad (16)
$$

Redundancy of elementary solutions

It follows from Eq.(16) that, if the dynamic quantities, Q and P, of forward propagation in time are known, the corresponding quantities for the reverse time solution can be easily deduced (and vice versa). The forward propagated solutions represent *physical responses*. Nevertheless, as later demonstrated, indirect quantities, such as the wavefront curvature (or its inverse), of the *non-physical*, time-reversed telescopic solutions can provide useful information about the local (interval) velocities. As a consequence, such solutions are bound to play an important role in velocity mapping, where the transformation is represented by tracing along normal or image rays. We observe, in passing, that these types of rays are telescopic in nature in a paraxial sense, since the wavefront is linear at the takeoff for an image ray (traced backward in time) and linear at the reflector for a normal ray (traced forward in time). In the next section we will focus on three elementary solutions, namely forward point source, as well as forward and backward telescopic in more detail (cf. Fig.1). The main purpose is to unravel the information each of them carries about the medium.

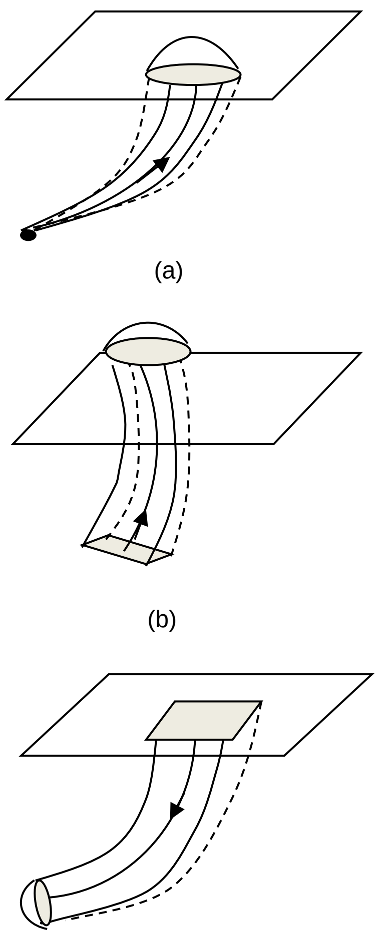

FIG.1 (a) Point-source response of the medium, (b) local plane-wave (telescopic) response of the medium and (c) time-reversed telescopic solution (non-physical solution but represents a mapping ray, here shown the image ray).

Medium response of elementary solutions

In the following, two different ray quantities will be computed for a general ray solution, namely the *geometrical spreading* $\mathfrak{I}$ and either *the time-wavefront curvature* M or *its inverse* M^{-1}.

The geometrical spreading is given as (Červený, 2001)

$$\mathfrak{I} = \frac{1}{v_f}\left|Q_f\right| , \tag{17}$$

and the time-wavefront curvature and its inverse can be calculated from the formulas (Červený, 2001)

$$M = P_f Q_f^{-1} \quad , \quad M^{-1} = Q_f P_f^{-1} . \tag{18}$$

The initial conditions in case of a *point-source* read (v_0 being the local (interval) velocity at source location)

$$Q_{ini} = 0 \quad , \quad P_{ini} = \frac{1}{v_0} , \tag{19}$$

and in case of a *telescopic* source

$$Q_{ini} = 1 \quad , \quad P_{ini} = 0 . \tag{20}$$

Point source solution (forward in time)

The point-source (*ps*) geometrical spreading is now given by the simple expression

$$_{\uparrow}\mathfrak{I}_{ps} = \frac{1}{v} T v_{rms}^2 , \tag{21}$$

where $v=v(0)$ represents the local (interval) velocity at the initial point (time 0) of the ray. Equation (21) can be regarded as a *generalization of the geometrical spreading formula* of Newman (1973) which is valid for a point source in a horizontally layered earth model.

A correspondingly simple expression can be found for $(M_{ps})^{-1}$

$$({}_{\uparrow}M_{ps})^{-1} = T v_{rms}^2 \left(1 - T\left\langle v^{-1} v_{qq} \right\rangle_w \right)^{-1} \cong T v_{nmo}^2 , \tag{22}$$

Equation (22) can also be considered as a generalization of the corresponding expression valid for a stratigraphic earth model. The relationship between rms-velocity and nmo-velocity in ray-centered coordinates follows directly from Eq.(22) reflecting their basic difference in case of smooth lateral velocity variations around the central ray.

Telescopic solution (forward in time)

Replacing the point source by a local telescopic (*ts*) condition, the geometrical spreading takes the form

$$_{\uparrow}\mathfrak{I}_{ts} = 1 - T\left\{ T v_{rms}^2 \left\langle v^{-1} v_{qq} \right\rangle - \left\langle v^{-1} v_{qq} \right\rangle_w \right\} \equiv 1 - \gamma \tag{23}$$

In Eq.(23), γ is an inhomogeneity factor which is zero if $v_{qq}=0$ (stratigraphic earth model), from which it immediately follows that $_{\uparrow}\mathfrak{I}_{ts} = 1$. Unlike the point-source case, the geometrical spreading is in general sensitive to v_{qq}.

In case of a telescopic solution, the time-wavefront curvature gives the proportional relationship with respect to medium parameters. It is given explicitly as

$$_{\uparrow}M_{ts} = -T\left\langle v^{-1}v_{qq}\right\rangle[1-\gamma]^{-1}. \quad (24)$$

In case of a stratigraphic earth model it follows directly from Eq.(24) that

$$_{\uparrow}M_{ts} = 0\,, \quad (25)$$

as expected.

Telescopic solution (time reversed)

This third elementary solution does not represent a physical wave, but plays a role in velocity mapping as discussed later. However, for the sake of completeness we state again the same two quantities as for the forward time propagated telescopic wave. The equivalences of Eqs.(23) and (24) are

$$_{\downarrow}\Im_{ts} = 1 - T\left\langle v^{-1}v_{qq}\right\rangle_w, \quad (26)$$

and

$$_{\downarrow}M_{ts} = T\left\langle v^{-1}v_{qq}\right\rangle\left\{1 - T\left\langle v^{-1}v_{qq}\right\rangle_w\right\}^{-1}. \quad (27)$$

Thus, direct comparison shows that no reciprocal relations exist between the two telescopic solutions for the geometrical spreading and the wavefront time-curvature.

Source-perturbed ray-propagator system

In this section we consider the case of paraxial tracing along a central ray where the receiver location is fixed but the source position is allowed to be perturbed very locally. Such a system can be used to study how a source perturbation manifests itself as corresponding time perturbations in the ray quantities. We will later see how useful this propagator system is when analyzing the velocity-stretch problem.

The source-perturbed, ray-propagator system can be constructed from Eqs.(4) and (9a-b). As shown in Fig.2, we consider two nearby point-source positions along the central ray. If the medium properties are assumed the same around both sources, the source-perturbed system can now be written as (linearized assumption)

$$\begin{bmatrix} _{\uparrow}dQ_f \\ _{\uparrow}dP_f \end{bmatrix} = {}_{\uparrow}\pi \begin{bmatrix} 0 & v_0^2 dT \\ -v_0^{-1}v_{0,qq}dT & 0 \end{bmatrix} \begin{bmatrix} _{\uparrow}Q_{ini} \\ _{\uparrow}P_{ini} \end{bmatrix} = \quad (28)$$
$$= \begin{bmatrix} _{\uparrow}Q_1 & _{\uparrow}Q_2 \\ _{\uparrow}P_1 & _{\uparrow}P_2 \end{bmatrix} \begin{bmatrix} 0 & v_0^2 dT \\ -v_0^{-1}v_{0,qq}dT & 0 \end{bmatrix} \begin{bmatrix} _{\uparrow}Q_{ini} \\ _{\uparrow}P_{ini} \end{bmatrix}.$$

In Eq.(28), the infinitesimal traveltime along the central ray between the two sources is assumed to be dT. Moreover, the local velocity quantities around the two sources are given by v_0 and $v_{0,qq}$. Finally, the propagator components $_{\uparrow}Q_1, _{\uparrow}Q_2, _{\uparrow}P_1, _{\uparrow}P_2$ are associated with the source corresponding to the shortest traveltime to the fixed receiver location.

Time-migration velocity stretching

The 2D diffraction curve employed in time-migration (referred to as a time-migration stacking moveout) is often represented in midpoint and half-offset domain *(y,h)*. Within a hyperbolic approximation it reads as follows (Hubral and Krey, 1980)

$$\tau^2(y,h) = \tau_0^2 + \frac{4}{v_M^2}\left[h^2 + (y-y_0)^2\right] \quad (29)$$

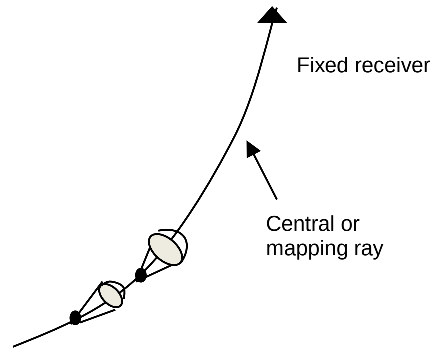

FIG.2 Schematics of a source-perturbed calculation. Two nearby point sources are initiated at the same firing time, the change in ray quantities with time evaluated at a fixed receiver location.

where $\tau_0 = \tau(y_0,0)$ is the two-way traveltime along the central ray, y_0 is the central midpoint and v_M denotes the migration velocity. We assume that the migration velocity, v_M, is already available, its actual determination being outside the scope of this paper. Equation (29) can be interpreted as a paraxial approximation of a diffraction traveltime surface that originates from an unknown depth point scatterer, D. More specifically, $\tau(y,h)$ approximates the traveltime of the diffraction ray that connects the source (of coordinate y-h) and the receiver (at coordinate y+h) on the seismic line to the scatterer at D. The point (y_0,τ_0) specifies the apex of the diffraction curve. As such, the image ray that starts at D hits the seismic line at y_0 and traveltime $\tau_0/2$. Equivalently, the image ray that propagates backward in time hits the scatterer D when the (one-way) traveltime $\tau_0/2$ is consumed (cf. Fig.3). We finally note that the wavefront along the image ray from D to y_0 (central image ray) is tangent to the seismic line at y_0, confirming that the slowness vector of the image ray is orthogonal to the measurement line also at y_0. In case of paraxial ray tracing (surrounding the image ray) from a point source at the true (unknown) scatterer location, D, the associated time-wavefront curvature M measured at y_0 can be calculated from Eq.(29)

$$M = \left.\frac{\partial^2(\tau(y_0,h)/2)}{\partial h^2}\right|_{h=0} = \frac{1}{v_M^2(\tau_0/2)}\,, \quad M^{-1} = v_M^2(\tau_0/2)\,. \quad (30)$$

From Eq.(22) it follows directly that $v_M = v_{nmo}$, where the nmo-velocity is calculated *along* the central/image ray. Equation (22) also shows how v_{nmo} relates to the actual medium velocities. In case of lateral velocity variations there is a stretch factor between v_{nmo} and v_{rms} (cf.Eq.(22)). This stretch factor is given by $_{\downarrow}\Im_{ts}$ (cf. Eqs.(26) and (36)). This implies that (with $\tilde{v}_{Dix}$ representing a generalized Dix velocity and v_0 being the local velocity around D)

$$\tilde{v}_{Dix} = \frac{d[(\tau_0/2)v_{nmo}]}{d(\tau_0/2)} = \frac{d[(\tau_0/2)(v_{rms}/{}_{\downarrow}\Im_{ts})]}{d(\tau_0/2)} \cong \frac{1}{{}_{\downarrow}\Im_{ts}}\frac{d[(\tau_0/2)v_{rms}]}{d(\tau_0/2)} = \frac{v_0}{{}_{\downarrow}\Im_{ts}} \qquad (31)$$

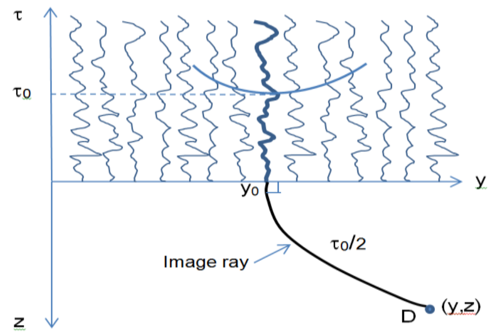

FIG.3 Constant-offset prestack time migrated section and the image ray concept.

The same result can be obtained more formally by considering the local time-variation of M^{-1}, namely (Cameron et al., 2007)

$$\frac{dM^{-1}}{d(\tau_0/2)} = \frac{d}{d(\tau_0/2)}\left[(\tau_0/2)v_M^2\right] = \tilde{v}_{Dix}^2 \ . \qquad (32)$$

For convenience, we introduce the one-way time-migration traveltime $\mathrm{T}=\tau/2$ Employing the perturbed ray system in Eq.(28) with point-source initial conditions, namely, $(Q_{ini}, P_{ini})^T = (0, 1/v_0)^T$ and $(Q_f, P_f)^T = (Q_2, P_2)^T$, yields

$$\begin{bmatrix} {}_{\uparrow}dQ_2 \\ {}_{\uparrow}dP_2 \end{bmatrix} = v_0^2 d\mathrm{T} \begin{bmatrix} {}_{\uparrow}Q_1 \\ {}_{\uparrow}P_1 \end{bmatrix} \quad . \qquad (33)$$

We are now in the position to compute the time perturbation $d(M_2)^{-1} = d({}_{\uparrow}Q_2, {}_{\uparrow}P_2^{-1})$

$$\begin{aligned} d(M_2)^{-1} &= d({}_{\uparrow}Q_2\, {}_{\uparrow}P_2^{-1}) = d({}_{\uparrow}Q_2)\, {}_{\uparrow}P_2^{-1} + {}_{\uparrow}Q_2\, d({}_{\uparrow}P_2^{-1}) \\ &= d({}_{\uparrow}Q_2)\, {}_{\uparrow}P_2^{-1} - {}_{\uparrow}Q_2\, d({}_{\uparrow}P_2)\, {}_{\uparrow}P_2^{-2} \ , \end{aligned} \qquad (34)$$

which, in view of Eq.(33), yields (at the migrated traveltime $T_0 = \tau_0/2$)

$$\begin{aligned} & d(M_2)^{-1} \cong v_0^2 d\mathrm{T}\left[{}_{\uparrow}Q_1 - {}_{\uparrow}Q_2({}_{\uparrow}P_2)^{-1}\, {}_{\uparrow}P_1\right] {}_{\uparrow}P_2^{-1} = v_0^2 d\mathrm{T}({}_{\uparrow}P_2)^{-2}, \\ & \Rightarrow \frac{d(M_2)^{-1}}{dT} \cong v_0^2\left({}_{\uparrow}P_2\right)^{-2} = v_0^2\left({}_{\downarrow}Q_1\right)^{-2} \quad , \end{aligned} \qquad (35)$$

in which the far most right equations follows from the symplectic condition in Eq.(6a), and where we also have made use of Eq.(16). Finally, the main result is obtained from Eqs.(32) and (35)

$$v_0 = {}_{\downarrow}Q_1 \tilde{v}_{Dix} \ . \qquad (36)$$

In Eq.(36) the proportionality factor between the generalized Dix velocity and the medium velocity is given by the geometrical spreading factor of a paraxial ray bundle surrounding the central/image ray and traced backward in (one-way) time $t_0/2$. Note that, in order to use this expression, we need to know the local/interval velocity along the ray (which is the unknown to be solved for in the beginning). In addition, we also need to calculate v_{qq} in order to carry out the tracing. Thus the velocity stretch problem is an inversion problem. The important result in Eq.(36) was first derived by Cameron et al. (2007).

Stacking (ZO) velocity stretching and CRS

By analogy with the previous section, we can make a revisit to the basic parameters associated with the Common Reflection Surface (CRS) method (see, e.g., Jäger et al., 2001). CRS uses as stacking moveout the generalized hyperbolic traveltime in offset-midpoint coordinates as a paraxial approximation around the (central) normal ray. In 2D, such moveout reads

$$t^2(y,h) = [t_0 + A(y-y_0)]^2 + B(y-y_0)^2 + Ch^2 \ , \qquad (37)$$

with

$$A = \frac{2\sin\beta}{v_0},\ B = \frac{2t_0\cos^2\beta(R_N)^{-1}}{v_0},\ C = \frac{2t_0\cos^2\beta(R_{Nip})^{-1}}{v_0}. \qquad (38)$$

In Eq.(38), β is the take-off angle of the central/normal ray, v_0 is the surface velocity and R_{Nip} is the radius of curvature (paraxial sense) associated with a point source at the normal-incidence point (NIP) of the central ray at the reflector segment and measured at the surface in ray-centered coordinates. Similarly, R_N is the radius of curvature of a local exploding reflector wave initiated around the same NIP. By considering data sorted in CMP gathers (i.e. with $y = y_0$), Eq.(37) reduces to the same form as the well-known NMO-equation:

$$t^2(y = y_0, h) = t_0^2 + Ch^2 \ . \qquad (39)$$

Within the CRS formulation Eq.(39) can be interpreted as a paraxial expansion of diffraction traveltimes that refer to a scatterer at NIP. From Eq.(39) it follows that (assuming *common-reflection-point at NIP*)

$$C = 2t_0\left(\frac{\partial^2[t(y_0,h)/2]}{\partial h^2}\Big|_{h=0}\right) = 2t_0 M_{NIP}\cos^2\beta = \frac{4}{(v_{nmo}/\cos\beta)^2} \qquad (40)$$

where $M_{NIP} = M_{ps}$ as given by Eq.(22) and the cosine factor represents coordinate transformation from ray centered system to Cartesian midpoint-offset system. Thus Eq.(40) represents *an effective medium representation valid for a smooth medium within a paraxial assumption.* Note that v_{nmo} is calculated *along* the (curved) normal ray and relates to the medium velocities through Eq.(22), similar to the time-migration case. From Eq.(40) it follows directly that (ray-centered coordinates)

$$(M_{NIP})^{-1} = v_0 R_{NIP} = v_{nmo}^2(t_0/2) \ . \qquad (41)$$

by analogy with Eq.(30) ($v_M = v_{nmo}$, with v_{nmo} associated with *image* ray). After stacking, the CRS-equation in Eq.(39) takes the (ZO) form ($h = 0$)

$$t^2(y,h=0)=[t_0+A(y-y_0)]^2+B(y-y_0)^2 \ . \qquad (42)$$

Optimal *parameters* A and B can now be obtained by stacking within the stack (in combination with a coherency measure such as semblance and proper choice of apertures). Equation (42) has a simple interpretation: the first term within the bracket represents the traveltime moveout in case a dipping reflector embedded in a model which is locally homogeneous along the direction orthogonal to the central ray direction and the second term is a correction term aiming to account for a possible curvature of reflector segment as well as lateral inhomogeneities in the velocity model. Note that $M_N = 1/(v_0R_N)$ *is not zero* in case of a plane reflector if lateral velocity variations exist around the central ray (cf. Eq.(24)). Thus after a completed CRS processing for a fixed trace (y_0), a time sample corresponding to a (one-way) traveltime $t_0/2$ will have three parameters attached: β, $M_{NIP} = (v_0R_{NIP})^{-1}$ and $M_N = (v_0R_N)^{-1}$. By analogy with the migration-velocity de-stretching a similar procedure can in principle be applied to a ZO section to de-stretch the NMO-velocities using the normal ray instead of the image ray as a central ray for the paraxial computations. However, differently from the time-migrated case, marching sample by sample along a fixed trace corresponds to a family of normal rays (as given by the takeoff angle β). By computing $d(M_{NIP})^{-1}/d(t_0/2)$ locally, and then combining Eqs.(35) and (41), the medium velocity can be inverted for as in the time-migration case. The counterpart of Eq.(36) then reads (note that the geometrical spreading now relates to the time-reversed tracing of an initially telescopic paraxial system with the normal ray serving as central ray)

$$v_0 = {}_{\downarrow}Q_1\sqrt{\frac{d(M_{NIP})^{-1}}{d(t_0/2)}} \equiv {}_{\downarrow}Q_1\tilde{v}_{DIX} \qquad (43)$$

However, unlike the time-migration case, one more parameter is now available namely M_N. In analogy with earlier analysis, we investigate what type of medium information is unwrapped if we calculate the quantity $dM_N/d(t_0/2)$ locally along a given stacked trace. If the local curvature can be neglected, this is straightforward by again employing the source-perturbed ray propagator system (but this time with a *telescopic initial condition*)

$$\begin{bmatrix} {}_{\uparrow}dQ_f \\ {}_{\uparrow}dP_f \end{bmatrix} = \begin{bmatrix} {}_{\uparrow}dQ_1 \\ {}_{\uparrow}dP_1 \end{bmatrix} = -v_0^{-1}v_{0,qq}dT\begin{bmatrix} {}_{\uparrow}Q_2 \\ {}_{\uparrow}P_2 \end{bmatrix}. \qquad (44)$$

The perturbation in the time-wavefront curvature M_N can be written as

$$dM_N \cong -v_0^{-1}v_{0,qq}dT\left[{}_{\uparrow}Q_2 - {}_{\uparrow}Q_1({}_{\uparrow}P_1)^{-1}{}_{\uparrow}P_2\right]({}_{\uparrow}P_1)^{-1} \qquad (45)$$

By again applying the symplectic condition it can be demonstrated that

$${}_{\uparrow}Q_2 - {}_{\uparrow}Q_1({}_{\uparrow}P_1)^{-1}{}_{\uparrow}P_2 = -({}_{\uparrow}P_1)^{-1} \ , \qquad (46)$$

which in combination with Eq.(45) can be used to establish the final result ($T = t_0/2$)

$$\frac{dM_N}{dT} \cong v_0^{-1}v_{0,qq}({}_{\uparrow}P_1)^{-2} = v_0^{-1}v_{0,qq}({}_{\downarrow}P_1)^{-2} \ , \qquad (47)$$

where we also have made use of Eq.(16). Note that the matrix ${}_{\downarrow}P_1$ is determined from the paraxial ray tracing around the normal ray with a telescopic initial condition and traced backward in time as discussed before. As in the time-migration case, we need to solve an inverse problem based on computing the paraxial mapping ray (here normal ray) system backward in time. Having access to both the telescopic and point-source responses of the earth model, introduces a further constraint on this inverse system (additional requirement that v_{qq} should fulfill Eq.(47)). However, if the local curvature of the reflector segment cannot be neglected, $dM_N/d(t_0/2)$ will interpret this as initial local lateral velocity variations (manifested in v_{qq}). The approach discussed here can be used as a refinement of a standard NIP tomography application (Duveneck, 2004). Using the NIP velocity field as input, further local refinement can be carried out within areas with good quality M_{NIP} (and possibly M_N) values. A further improvement in building smooth consistent medium velocities in depth will be to simultaneously work with prestack time-migrated and CRS type of stacked data.

Conclusions

An incremental ray-propagator system has been introduced as a possible tool to more easily analyze the medium information carried by different elementary earth responses (point and telescopic type). This system is not to be regarded as a practical paraxial ray-tracing system, but serves the purpose of an analyzing tool. The concept is also well tailored for analyzing the so-called velocity stretch problem, i.e. how to recover local (interval) velocities from either time-migration or stacking velocities. Finally it should be noted that the generalization to the 3D case is rather straightforward.

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